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Thomas Bayes's Work on Infinite Series

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Apart from his published letter on the divergence of the series for log $z!$, Thomas Bayes left some unpublished material on infinite series. In this paper this material, consisting of passages in a Notebook and a letter to John Canton, is examined and related to that in the published work. The series discussed in the published letter is found to have received extensive investigation in the Notebook, and it is suggested that this investigation perhaps made Bayes aware of the divergence of the series for log $z!$. © 1991 Academic Press, Inc.

Außer dem veröffentlichten Brief über die Divergenz der Reihe für log $z!$ enthält der Nachlaß von Thomas Bayes unveröffentlichtes Material zu unendlichen Reihen. Dieses Material, das aus Passagen in einem Notizbuch und einem Brief an John Canton besteht, soll in dieser Arbeit untersucht und in Beziehung zu den veröffentlichten Arbeiten gebracht werden. Es stellt sich heraus, daß die in dem veröffentlichten Brief besprochene Reihe im Notizbuch ausführlich untersucht wurde, und der Schluß wird gezogen, daß Bayes eventuell durch diese Untersuchung auf die Divergenz der Reihe für log $z!$ aufmerksam geworden ist. © 1991 Academic Press, Inc.

A part son oeuvre publiée sur la divergence de la série pour log $z!$, Thomas Bayes a laissé des matériaux inédits sur les séries infinies. Dans cet article, ces matériaux—des extraits d'un bloc-notes et une lettre à John Canton—sont examinés et mis en rapport avec l'oeuvre publiée. La série discutée dans la lettre publiée a été longuement examinée dans le bloc-notes et il est suggéré ici c'est peut-être par cet examen que Bayes a pris conscience de la divergence de la série pour log $z!$. © 1991 Academic Press, Inc.

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1. INTRODUCTION

Although chiefly, and correctly so, remembered for his work in probability (his posthumous "Essay towards solving a problem in the doctrine of chances" of 1763—published in 1764—played a crucial rôle in the development of the vigorous modern school of Bayesian statistics), Thomas Bayes (1702?–1761) made other important contributions to mathematics. Chief among these was his *Introduction to the Doctrine of Fluxions* of 1736. This treatise, a possible cause [1] of his election to a fellowship in the Royal Society in 1742, has been the subject of a careful study by G. C. Smith [1980], and we need accordingly pay no further attention to it here.

However, the same issue of the *Philosophical Transactions* in which Bayes's "Essay" appeared also carried a letter from him on infinite series [Bayes 1763b].

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In the preceding half-century [2] this topic had been no stranger to this august journal, which had carried papers of particular relevance [3] by A. de Moivre [1714], J. Dodson [1753], J. Eames [1736], J. Landen [1760], T. Simpson [1748, 1751, 1755, 1758], and B. Taylor [1717]. Yet Bayes's "Letter to Canton" was in a sense unique, addressing itself, as we shall see later in this paper, to the question of the divergence of a certain series.

This is the only published work by Bayes on infinite series *per se,* although such series are used in the evaluation of the incomplete beta function (see Bayes [1763a, 1764]). Yet other fragments dealing with this matter remain: part of a manuscript letter from Bayes to John Canton (of the Royal Society) on Simpson's work, and several pages in a Notebook [4] that has been attributed to Bayes [5] (see Section 5 of the present paper for reasons for this attribution). If the published "Letter to Canton" is seen as a finished product, in some sense, it would not be without interest to search the Notebook for anything pertinent to the subject matter of the published letter [Bayes 1763b]. It is to such an investigation that the main body of this paper is devoted. We shall see that there is extensive discussion of series for $\log z!$ in the Notebook, and that, in the course of his work, Bayes calculated sufficiently many coefficients in the series to become aware of its divergence.

2. THE MANUSCRIPT FRAGMENT TO CANTON

This anonymous letter (or fragment thereof) bears neither date nor salutation; it is merely addressed "Sir." The handwriting is markedly similar to that in the Notebook, so it is likely that the same person was responsible for both. In Section 5 evidence will be given for attributing both documents to Bayes.

The letter, now among the Canton papers in the Royal Society, opens as follows:

> You may rem. a few days ago we were speaking of M[r] Simpson attempting to show the great advantage of taking the mean between several astron. observations rather than trusting to a single observation carefully made in order to diminish the errors arising from the imperfection of instruments and the organs of sense. [Bayes undated—a]

This letter then clearly refers to the matter considered by Simpson in his paper [Simpson 1755], and dealt with rather more fully in his tract [Simpson 1757]. Bayes recalls that he and the addressee had agreed that the first method quoted above "was undoubtedly the best upon the whole," but he suggests that "M[r] Simpson has not justly represented its advantage: neither is it by far so great as he seems to make it" [Bayes undated—a].

As an example Bayes considers the following:

> if a single observatiō may be relied on to 5″, & you take the mean of six observations it is above 5000 to 1 that your conclusion do's not differ 3″ from the truth, & by sufficiently increasing the number of observations you may make it as probable as you please that the result does not differ from the truth above a single second or any small quantity whatsoever. [Bayes undated—a]

This is exactly (a part of) the example considered by Simpson, and its citation in this letter suggests strongly that the latter was written after Bayes had seen Simpson's paper—perhaps shortly after it was read before the Royal Society on April 10, 1755.

In fact the ratio of 5000 to 1 mentioned by Bayes is not given by Simpson. The example considered by the latter runs as follows:

> I shall suppose here, that every observation may be relied on to 5 seconds; and that the chances for the several errors, $-5''$, $-4''$, $-3''$, $-2''$, $-1''$, $-0''$, $+1''$, $+2''$, $+3''$, $+4''$, $+5''$, included within the limits thus assigned, are respectively proportional to the terms of the series 1,2,3,4,5,6,5,4,3,2,1. [Simpson 1755, 91].

It follows that the chance of an error in $\{-3'', -2'', \ldots, +3''\}$ is 30/36 or 5/6; and although this is not mentioned by Simpson, he does say that "the chance for an error exceeding 3 seconds, will not be 1/1000 part so great from the Mean of six, as from one single observation" [Simpson 1755, 92], from which the ratio cited by Bayes then follows immediately.

Neither Bayes nor Simpson explicitly derived the ratios mentioned above. However, Simpson considered in detail the derivation of the ratio when the error is not more than $1''$. The problem is described in his second proposition as follows:

> Supposing the respective chances, for the different errors which any single observation can admit of, to be expressed by the terms of the series $r^{-v} + 2r^{1-v} + 3r^{2-v} + \cdots + \overline{v+1}r^{0} \cdots + 3r^{v-2} + 2r^{v-1} + r^{v}$ (whereof the coefficients, from the middle one $(v + 1)$, decrease, both ways, according to the terms of an arithmetical progression): 'tis proposed to determine the probability, or odds, that the error, by taking the Mean of a given number (t) of observations, exceeds not a given quantity (m/t). [Simpson 1755, 87]

(In Proposition I a geometric progression is assumed.) It is shown that, when the chance for an error in excess is equal to that of an error in defect, "the sum of the chances for all the inferior numbers (inclusive)" (i.e., all numbers less than or equal to some given number p) is

$$\frac{p}{1}\frac{(p-1)(p-2)}{2\quad\ 3}\frac{(p-3)}{4}(n) - \frac{p'}{1}\frac{(p'-1)}{2}\frac{(p'-2)}{3}\frac{(p'-3)}{4}(n) \times n$$
$$+ \frac{p''}{1}\frac{(p''-1)}{2}\frac{(p''-2)}{3}\frac{(p''-3)}{4}(n) \times \frac{n}{1}\cdot\frac{n-1}{2}$$
$$- \frac{p'''}{1}\frac{(p'''-1)}{2}\frac{(p'''-2)}{3}\frac{(p'''-3)}{4}(n) \times \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\ \&\text{c.},$$

where $n = 2t$, $p' = p - w$, $p'' = p - 2w$, $p''' = p - 3w$ and $w = v + 1$. Denoting by D the difference between the above series and $w^n/2$, Simpson concluded that

> $D/(\frac{1}{2}w^n)$ will be the true measure of the required probability, that the error, by taking the Mean of t observations, exceeds not the quantity m/t, proposed. [Simpson 1755, 90]

Setting $v = 5$, $t = 6$, we have $n = 2t = 12$, $w = v + 1 = 6$, $p = tv + n + m = 42 + m$. Simpson solved for m from $m/t = \pm 1$, and declared that the negative sign is "the most commodious." Thus $m = -6$. Substitution in the series given above yields the value 299,576,368 which, when subtracted from $6^{12}/2$, gives $D =$ 788,814,800. Simpson then remarked:

> Therefore the required probability, that the error, by taking the Mean of six observations, exceeds not a single second, will be truly measured by the fraction 788814800/1088391168; and consequently the odds will be as 788814800 to 299576368, or as $2\frac{5}{8}$ to 1, nearly. But the proportion, or odds, when one single observation is relied on, is only as 16 to 20, or as $^{8}/_{10}$ to 1. [Simpson 1755, 92]

In the fragment Bayes took exception to the suggested possibility of the constant increase in probability by increasing numbers of observations. Indeed, he wrote

> Now that the Errors arising from the imperfection of instrum^{ts}. & the organs of sense should be thus reduced to nothing or next to nothing only by multiplying the number of observations seems to me extremely incredible. On the contrary the more observations you make with an imperfect instrum^{t}. the more certain it seems to be that the error in your conclusion will be proportional to the imperfection of the instrum^{t}. made use of. [Bayes undated—a]

For if not, he went on, what would be the advantage in using an accurate instrument rather than an imperfect one to effect the observations?

Since Bayes could find no error in Simpson's calculations, he concluded that the latter's hypothesis must be wrong. He suggested that the chances of errors of excess or defect should *not* be regarded as the same. It is this sameness which leads to the great advantage Simpson found in his method, for if the chances of errors in excess were greater than those in defect, "by taking the mean of many observations I shall only more surely commit a certain error in excess." It thus follows that, in many cases, repeated observations on an imperfect instrument and the subsequent taking of the mean may not necessarily result in a diminishing of the error.

It appears (see Stigler [1986, pp. 88–98] that Bayes's views were communicated to Simpson, for there were several changes in the latter's tract of 1757.

3. THE PUBLISHED "LETTER TO CANTON"

The message of the published letter [Bayes 1763b] is succinct: it has sometimes been asserted that, for any natural number z,

$$\sum_{1}^{z} \log k = \frac{1}{2} \log C + \left(z + \frac{1}{2}\right) \log z - S \tag{1}$$

where $C = 2\pi$ and

$$S = z - \frac{1}{12z} + \frac{1}{360z^3} - \frac{1}{1260z^5} + \frac{1}{1680z^7} - \frac{1}{1188z^9} + \&c. \tag{2}$$

Not so! says Bayes. For although it is true that the right-hand side of (1) approaches the value of the sum as z increases (and provided that an appropriate number of terms are taken), the fact is that after the fifth term in the right-hand side of (2), the coefficients begin to increase, and so "the whole series can have no ultimate value whatsoever" [Bayes 1763b, 271]. Indeed the coefficients of the nth and $(n + 1)$th terms on the right-hand side in (2)—call them u_n and u_{n+1}—satisfy $u_{n+1}/u_n > (n - 1)(2n - 3)/(6n + 3)$ for $n \geq 3$. (Bayes gives this result in slightly different form. For discussion of a bound given by Legendre see C. Tweedie [1922, 44]). Bayes also took note of various special cases (which of course also fail) but noted that "one instance is sufficient to shew that those methods are not to be depended upon, from which a conclusion follows that is not exact" [Bayes 1763b, 271].

The Stirling–de Moivre series "for" log $z!$ was of course well known (see R.C. Archibald [1926] and Tweedie [1922] for a discussion). However, it appears that neither de Moivre nor Stirling had appreciated the mere symbolism of the series: not having continued the series in (2) far enough, although the rule for continuance was known, they failed to spot the divergence (see Hardy [1949] for a discussion of such series). As W. Edwards Deming [1963] notes, Euler, some six years before the death of Bayes, had noted the divergence of the factorial series for $z = 1$ in Sections 157–159 of the second part of his *Institutiones calculi differentialis* [1755]. Here, from the expression

$$\sum_{1}^{x} \log k = x \log x - x + \frac{1}{2} \log x + \frac{A}{1.2x} - \frac{B}{3.4x^3} + \frac{C}{5.6x^5} - \frac{D}{7.8x^7} + \text{etc.} + \text{Const.},$$

where A, B, C, D are Bernoulli numbers, Euler concludes that the constant, found on setting $x = 1$, is given by

$$\text{Const.} = 1 - \frac{A}{1.2} + \frac{B}{3.4} - \frac{C}{5.6} + \frac{D}{7.8} - \text{etc.},$$

"which series," he then noted, "is, on account of excessive divergence, unsuitable for the obtaining of the value of [the constant] at all accurately" [Euler 1755]. (The determination, not merely of an approximate value, but of the true value, is obtained by using Wallis's product for $\pi/2$ in Section 158) [6]. It is, however, uncertain as to whether Euler appreciated that the series would diverge for *all* values of x, no matter how large.

Thus it may well be that Bayes was the first to note the asymptotic nature of the relationship between log $z!$ and the right-hand side of (1). But was he perhaps drawn to this conclusion by some knowledge of Euler's work? To see whether anything at all definite can be said on this point let us turn to the Notebook to see whether any clues can be found there.

4. THE NOTEBOOK

The table of contents of the Notebook contains the following pertinent entry:

Pag. primae novem—pertinent ad problemata Data area invenire summā aequidistantium ordinatorū & vice versa sive quod eodem redit Data fluxione invenire incrementū aut vice versa. 10. Invenire Log, $\overline{\frac{z+1}{z}}\Big|^{z+1/2}$ ejusque integrale. [Bayes undated—b]

(That is, "the first nine pages pertain to the problem: Given the area to discover the greatest of the equidistant ordinates & vice versa, or, what reduces to the same thing, Given the fluxion to discover the increment or vice versa. 10. To find $\log \left(\frac{z+1}{z}\right)^{z+1/2}$ and its integral.")

It is in these first ten pages that we find results on infinite series which have a direct bearing on Bayes's published "Letter to Canton" on $\log z!$.

Fundamental to this work is the following result of Section 1:

Let t be a uniformly flowing quantity $\underset{\cdot}{t} = \dot{t} = 1$

$$x = N + at + \frac{bt^2}{2} + \frac{ct^3}{2.3} + \frac{dt^4}{2.3.4} + \&\text{c.} + \frac{kt^n}{2.3.4.5.\&\text{c}.n}$$

&

$$\underset{\cdot}{x} = \dot{x} + \frac{\ddot{x}}{2} + \frac{\dddot{x}}{2.3} + \frac{\ddddot{x}}{2.3.4} + \&\text{c.}$$

as will be evident by find[ing] $\underset{\cdot}{x}$ by the method of fluxions. Also in the same manner

$$\dot{x} = \underset{\cdot}{x} - \underset{\cdot\cdot}{x}/2 + \underset{\because}{x}/3 - \underset{::}{x}/4 + \underset{\substack{::\\ \cdot}}{x}/5 - \&\text{c}$$

And thus also the relation between $\ddot{x}$ and $\underset{\cdot\cdot}{x}$ & so on may be found.

(Here letters which are "pricked" above and below denote fluxions (or derivatives) and finite differences respectively.) The expression for $\dot{x}$ may be found as follows. From the expression given for $\underset{\cdot}{x}$ we have, by transposition,

$$\dot{x} = \underset{\cdot}{x} - \frac{\ddot{x}}{2} - \frac{\dddot{x}}{2.3} - \frac{\ddddot{x}}{2.3.4} - \&\text{c.}; \tag{3}$$

it also follows from the expression for $\underset{\cdot}{x}$ that

$$\underset{\cdot\cdot}{x} = (\underset{\cdot}{x})^{\cdot} + \frac{1}{2}(\underset{\cdot}{x})^{\cdot\cdot} + \frac{1}{2.3}(\underset{\cdot}{x})^{\therefore} + \frac{1}{2.3.4}(\underset{\cdot}{x})^{::} + \&\text{c.},$$

where $(\underset{\cdot}{x})^{\cdot}$ denotes the fluxion of $\underset{\cdot}{x}$, etc. Substitution of the given expression for $\underset{\cdot}{x}$ in this last series yields

$$\underset{\cdot\cdot}{x} = \left(\ddot{x} + \frac{\dddot{x}}{2} + \cdots\right) + \frac{1}{2}\left(\dddot{x} + \frac{\ddddot{x}}{2} + \cdots\right) + \cdots.$$

Substitution in (3) of $\ddot{x}$ obtained from this expression gives

$$\dot{x} = \underset{\cdot}{x} - \underset{\cdot\cdot}{x}/2 + f(\dddot{x}, \ddddot{x}, \ . \ . \ .).$$

This process may then be repeated.

In the first two pages Bayes derived, using this result, what is essentially the Euler–MacLaurin sum formula. This he stated as follows:

if upon the base at equal distances each $= z = 1$ you erect any number of ordinates, call the 1st x & the last y the area between the 1st and the last A & the sum of all the ordinates except the last S. that

$$S = A - \frac{\dot{A}}{2} + \frac{\ddot{A}}{12} - \frac{\overset{::}{A}}{720} + \frac{\overset{:::}{A}}{30240} - \frac{\overset{::::}{A}}{1209600} + \&c.$$

& $\dot{A} = y - z$ [should be $y - x$]. But note that there are some exceptions to this rule.

In fact, on writing the Euler–MacLaurin sum formula as

$$\sum_1^n f(m) \sim \int_a^n f(x)dx + C + \frac{1}{2}f(n) + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{B_r}{(2r)!} f^{(2r-1)}(n),$$

where the B_r are the Bernoulli numbers [7], we find that the sum and the integral given here correspond respectively to Bayes's S and A, with $C = -f(n)$. This result is then used in Section 3 to deduce that

$$\log\left(\frac{z+1}{z}\right) = \frac{2}{2z+1}\left[1 - \frac{1}{12}\left(\frac{1}{z+1} - \frac{1}{z}\right) + \frac{2!}{720}\left(\frac{1}{(z+1)^3} - \frac{1}{z^3}\right) + \frac{4!}{30240}\left(\frac{1}{(z+1)^5} - \frac{1}{z^5}\right) + \cdots\right]$$

(The "+ · · ·" is not in fact given by Bayes, but its presence seems indicated by his "&c." and the Euler–MacLaurin sum formula.) This result we shall write for later reference in the form

$$\left.\begin{aligned} \log\left(\frac{z+1}{z}\right)^{z+1/2} &= 1 + \sum_{r=1}^{\infty} (-1)^r \frac{B_r}{(2r-1)(2r)} \Delta(1/z^{2r-1}) \\ &= 1 + \sum_{r=1}^{\infty} (-1)^r \frac{B_r}{(2r)!} \Delta((1/z)^{(2r-2)}) \\ &= 1 + \sum_{r=1}^{\infty} (-1)^r \frac{B_r}{(2r)!} [\Delta(1/z)]^{(2r-2)} \end{aligned}\right\}, \tag{4}$$

where $\Delta f(z) = f(z+1) - f(z)$ and $f^{(n)}(z)$ denotes the nth derivative of $f(z)$ [8].

The next pages of the Notebook contain (partly in shorthand) an extract from Sections 837, 839, 842, and 847 (concluding with a very brief extract from Section 827) of MacLaurin's *A Treatise of Fluxions* of 1742, in which series are given for

(i) $\log(m + z) - \log m$, when m is given;
(ii) various formulae derived from (i);
(iii) $\sum_1^{n-1} \log k$;
(iv) series for $\varepsilon N/(N - 1)$ and $\varepsilon N/(N^2 - 1)$, where N is the number whose hyperbolic logarithm is ε.

On page 7 of the Notebook the following problem is considered:

Sit $z = \log, x$ & ex data equatione $x = 1 + \dfrac{2z}{2v - z}$ invenire v ex data z. (5)

(That is, "Let $z = \log x$ and from the given equation $x = 1 + 2z/(2v - z)$, find v for given z.")

Now on the face of it this seems a rather curious problem. However, on solving for v we find that

$$v = \frac{z(x+1)}{2(x-1)} = \frac{x+1}{2(x-1)} \log x.$$

Setting $x = (t+1)/t$ and writing $v \equiv v(x)$ as a function of t, we find that $v(t) = (t + 1/2)\log(t + 1/t)$. But from $v = z(x+1)/2(x-1)$, and on recalling that $z = \log x$, we see that we can write $v \equiv v(z)$ as

$$v(z) = \frac{z(e^z+1)}{2(e^z-1)} = \frac{ze^z}{e^z-1} - \frac{z}{2},$$

the first term on the right-hand side here being the first of the series mentioned in (iv) above—and in fact this is a series which generates the Bernoulli numbers.

Using the above expression for $v(t)$ we find that, for $n \in N$,

$$\begin{aligned}\sum_1^n v(t) &= \sum_1^n \log\left(\frac{t+1}{t}\right)^{t+1/2}\\ &= \left(n+\frac{1}{2}\right)\log(n+1) - \sum_1^n \log k.\end{aligned}$$

Thus

$$\begin{aligned}\log n! &= \sum_1^n \log k\\ &= \left(n+\frac{1}{2}\right)\log(n+1) - \sum_1^n v(t)\\ &= \left(n+\frac{1}{2}\right)\log(n+1) - \sum_1^{n-1} v(t) - v(n)\\ &= \left(n+\frac{1}{2}\right)\log(n+1) - \sum_1^{n-1} v(t) - \log\left(\frac{n+1}{n}\right)^{n+1/2}\\ &= \left(n+\frac{1}{2}\right)\log n - \sum_1^{n-1} v(t).\end{aligned}$$

Thus, if one is interested in series for $\log n!$, $v(\cdot)$ is an eminently reasonable function to consider. Indeed, we shall see in what follows that Bayes gave expansions not only for $v(\cdot)$, but also for $\log n$ and $\Sigma_1^n \log k$.

Bayes also used the equation $x = 1 + 2z/(2v - z)$ to deduce the coefficients in the series for $(z + \frac{1}{2}) \log(1 + 1/z)$. Setting $\dot{z} = 1$ and using $z = \log x$, he found that $1 = \dot{x}/x$. Differentiation of the initial expression for x and the equating of the result to x then yields

$$v^2 + \dot{v}z = v + z^2/4. \tag{6}$$

Bayes now assumed that

$$v = 1 + az^2 - bz^4 + cz^6 - dz^8 + ez^{10} - fz^{12} + \&c.,$$

and substitution of this expression, together with $\dot{v}$, in (6) resulted in an identity from which the coefficients $a,b,c,$ . . . can be found. Bayes, in fact, gave the coefficients, up to that of the term in z^{14}, as

$$a = 1/12, \quad b = 1/720, \quad c = 1/30240,$$

$$d = 1/1209600, \quad e = 1/47900160,$$

$$f = 691/2^{11}.3^6.5^3.7^2.11.13,$$

$$g = 1147/2^9.3^8.5^4.7^2.11.13.$$

These are also given as

$$a = 1/12, \quad 2!b = 1/360, \quad 4!c = 1/1260,$$

$$6!d = 1/680, \quad 8!e = 1/1188,$$

$$10!f = 691/360360, \quad 12!g = 2294/61425,$$

where we have replaced Bayes's products by factorials.

That all these coefficients were carefully evaluated suggests that Bayes was aware of the behaviour of the coefficients with increasing powers of z, a behaviour to which, as we have already seen, he drew attention in his "Letter to Canton."

Only one expansion is given for log $(z - 1)$, one which we can write in an admittedly anachronistic notation as

$$\log(z - 1) = \Delta f(z - 1) - \frac{1}{2}\Delta^2 f(z) + 2\left(\frac{1}{3}S - \frac{1}{4}\Delta S + \frac{1}{5}\Delta^2 S - \cdots\right) \tag{7}$$

where $f(z) = \log z^z$ and $S = \frac{1}{2}\Delta(1/z) + \frac{1}{12}\Delta(1/z^3) + \frac{1}{30}\Delta(1/z^5) + \cdots$ This Bayes derived by writing

$$1/(z - 1)^2 = -\Delta(1/(z - 1)) + \frac{1}{2}\Delta^2(1/(z - 1)) - \frac{1}{3}\Delta^3(1/(z - 1)) + \cdots \tag{8}$$

and then by noting that, since

$$\Delta^2(1/(z - 1)) = 2/z^3 + 2/z^5 + 2/z^7 + \cdots,$$

it follows that

$$\begin{aligned}\Delta^3(2/(z - 1)) &= \Delta(\Delta^2(1/(z - 1)))\\ &= \Delta(2/z^3 + 2/z^5 + 2/z^7 + \cdots)\\ &= 2/z^3 - 2/(z + 1)^3 + \cdots\end{aligned}$$

and so on. Then, on taking (8) and "summendo fluentes bis" (i.e., by integrating twice) Bayes arrived at (7). Note, incidentally, that no arbitrary constants enter into this integration.

Immediately after this derivation Bayes found a series for $\Sigma \log(z - 1)$, preceding the actual series with the words "Quare integrale logarithmi τoυ $z - 1$." (That is, "Whereby the sum of the logarithm of $z - 1$ is.") This is really equivalent to saying that $\Delta^{-1} \log z = \log \Gamma(z) + c$. It is clear from what follows that the operation concerned here is in fact one which is inverse to that of finite difference. Once again, though, no arbitrary constants appear. The series given is

$$\log(z-1)^{z-1} - \frac{1}{2}\,\Delta\log(z-1)^{z-1} + \frac{1}{3}\left[\frac{1}{z} + \frac{1}{6z^3} + \frac{1}{15z^5} + \frac{1}{28z^7} + \cdots\right]$$
$$- \frac{1}{4}\left[\Delta(1/(z-1)) + \frac{1}{6}\,\Delta(1/(z-1)^3) + \frac{1}{15}\,\Delta(1/(z-1)^5) + \cdots\right]$$
$$+ \frac{1}{5}\left[\Delta^2(1/(z-1)) + \frac{1}{6}\,\Delta^2(1/(z-1)^3) + \frac{1}{15}\,\Delta^2(1/(z-1)^5) + \cdots\right] + \cdots.$$

(There is a slip here in the original, the second differences being given as third differences.) This series is rewritten at the top of p. 7 of the Notebook as

$$K + z + \log(z-1)^{(3z-3)/4} - \frac{1}{2}\log z^z + x/3 - x/4 + x/5 - x/6 + \&\text{c}.$$

where $x = 1/z + 1/6z^3 + 1/15z^5 + \cdots$.

The presence of the terms $K + z$ here requires some comment. It would appear that, in accordance with the custom of his time, Bayes adopted a somewhat cavalier attitude to constants of integration (cf. Simpson's *The Doctrine and Application of Fluxions* [1823, Vol. 1, Sect. 6]). Thus while no constant appears on passing from $\dot{y} = \dot{x}/x$ to $y = \log x$, one further integration yields a constant which in turn, on being operated upon by Δ^{-1}, yields $K + z$, or more strictly $K + cz$.

Five series are given in the notebook for $(z + 1/2)\log(z + 1/z)$. Written anachronistically, they are the following:

S1. $1 - \dfrac{1}{12}\,\Delta(1/z) + \dfrac{1}{360}\,\Delta(1/z^3) - \dfrac{1}{1260}\,\Delta(1/z^5) + \cdots$

S2. $1 - \dfrac{1}{3.4}\,\Delta^3(\log z^z) + \dfrac{2}{4.6}\,\Delta^4(\log z^z) + \cdots + \dfrac{n}{(n+2)(2n+2)}$
$\Delta^n(\log z^z) + \cdots$

S3. $1 - \dfrac{1}{12}\,\Delta(1/z) + \dfrac{1}{120.3!}\,\Delta^3(1/z) - \dfrac{1}{30.5!}\,\Delta^5(1/z) + \cdots$

S4. $1 - \dfrac{1}{12}\,\Delta(1/z) - \dfrac{1}{10.12}\,[\Delta(1/z)]^2 + \dfrac{1}{7.10.12}\,[\Delta(1/z)]^3 + \cdots$

S5. $1 - \sum\limits_{r=1} (-1)^r B_r z^{2r}/(2r)!$

(S5 is, in fact, given as a series for $v(z)$, a series which Bayes obtained by the assumption that $v(z) = 1 + az^2 - bz^4 + cz^6 - \cdots$ and by appropriate differentiation of the original expression (5) in v, z and x. Note that this is $\varepsilon/2$ plus the series for MacLaurin's $\varepsilon N/(N - 1)$ mentioned in (iv) above.) The series S2 does not seem particularly useful. The series S1, S3, and S4, the series in powers of $1/z$, can be reconciled by using known properties of the difference operator (compare the series for $(z + 1/2) \log(z + 1/z)$ given earlier in this section).

Finally, on the tenth page of his Notebook Bayes gave a series for

$$\sum \log \left(\frac{z + 1}{z}\right)^{z+1/2},$$

prefacing it again by the word "integrale" and deducing it by applying this operation to S3.

5. CONCLUSION

It is almost impossible to date the Notebook, except in a very broad way. Many of the entries are undated: the earliest work cited is Roger Cotes's *Harmonia Mensurarum* of 1722, the latest is a paper by T. Allen in *The London Magazine, or Gentleman's Monthly Intelligencer* of 1760. Many of the works cited in the Notebook were published in several editions, and it is not possible to say which of these editions were used by Bayes. The left-hand side of p. 3 of the Notebook starts off with the words "M^r M^{c}Laurin says. 837," and the entry that follows is in fact from paragraph 837 of Colin MacLaurin's *Treatise of Fluxions* of 1742. Except for other references to this book, none of the rest of the work on series is dated.

It is not possible categorically to aver that the Notebook is by Thomas Bayes. However, there are four reasons which make this attribution likely, viz.

(i) The characteristic writing. Indeed, it was on this ground that the attribution, inscribed by M. E. Ogborn of the Equitable Life Assurance Society on the first page of the Notebook, was originally made. The handwriting in the Notebook accords well with that of the fragment of the letter to Canton and with that of some notes on electricity, both of these latter manuscripts being in the Canton papers of the Royal Society. Unfortunately, however, neither of these documents is itself definitely known to be by Bayes, but the handwriting is markedly similar to that of the published letter on series.

(ii) The characteristic shorthand. The system used is (a slight adaption of) one proposed by Elisha Coles in 1674, this in turn being an adaption of an earlier one given by Thomas Shelton in his *Zeiglographia* of 1654.

(iii) A passage on probability. The notebook contains a proof of the second rule published in Bayes's *Essay*.

These reasons have been noted in print before—see [Home 1974–1975] for the first two and [Dale 1986] for the third. I believe that the matters discussed in the present paper furnish a fourth reason. All the work in the first ten pages of the

Notebook is concerned with a subject in which Bayes is known to have been interested, and while there is no direct statement here of the matter discussed in his published *Letter,* it is, I suggest, quite possible that Bayes noticed the divergence (or even the asymptotic nature) of the series for log $z!$ in the course of his work on the computation of the coefficients. However, to answer a question raised earlier (see Section 3), there is no evidence of Bayes's being acquainted with Euler's work on the series for log $z!$, and the identification of the divergence of the series for general values of z may be ascribed to Bayes.

APPENDIX: A SOURCE OF INFORMATION AS TO BAYES'S EDUCATION

Where Thomas Bayes was educated has long been unknown (see, for example, Holland [1962]). It has recently been discovered, however, that he spent some time at Edinburgh University (see Dale [1989]), and it is our aim here to list all the relevant information that has been found in the records of that institution.

In the early eighteenth century several letters passed between English dissenting ministers (among them Christopher Taylor and Benjamin Bennet) and William Carstares, then principal of Edinburgh University, containing proposals to attract sons of dissenters to Edinburgh as students (see Grant [1884 II, 262] for further details). It was possibly as a result of this initiative that we find Thomas Bayes, Edmund Calamy, John Horsley, Isaac Maddox, and Skinner Smith, among others, enrolled as members of the College of James the Sixth.

The complete list (as far as has been ascertained) of references to Bayes in the archives of Edinburgh University, in no particular order, runs as follows (the references in square brackets are the shelf-marks of the university's special collections department):

1. [Da]. *Matriculation Roll of the University of Edinburgh. Arts–Law–Divinity. Vol. 1, 1623–1774. Transcribed by Dr. Alexander Morgan, 1933–1934.* Here, under the heading "Discipuli Domini Colini Drummond qui vigesimo-septimo die Februarii, MDCCXIX subscripserunt", we find the signature of Thomas Bayes—a signature, by-the-by, remarkably similar to that in the records of the Royal Society. This list contains the names of 48 students of Logic.

2. [Da.1.38] *Library Accounts 1697–1765.* Here, on the 27th February 1719, we find an amount of £3-0-0 standing to Bayes's name—and the same amount to Horsley, Maddox, and Smith. All of these, incidentally, are listed under the heading "supervenientes," i.e., "such as entered after the first year, either coming from other universities, or found upon examination qualified for being admitted at an advanced period of the course" [Dalzel 1862 II, 184].

3. [Da.2.1] *Leges Bibliothecae Universitatis Edinensis. Names of Persons admitted to the Use of the Library.* The pertinent entry here runs as follows:

> Edinburgi decimo-nono Februarij Admissi sunt hi duo Juvenes praes. D. Jacobo Gregorio Math. P. Thomas Bayes. Anglus. John Horsley Anglus.

Unfortunately no further record has been traced linking Bayes to this eminent mathematician.

4. [Dc.5.24[2]]. In the *Commonplace Book of Professor Charles Mackie,* we find, on pp. 203–222, an *Alphabetical List of those who attended the Prelections on History and Roman Antiquitys from 1719 to 1744 Inclusive. Collected 1 July, 1746.* Here we have the entry

Bayes ( ), Anglus. 1720, H. 21,H. 3

The import of the final "3" is uncertain.

5. *Lists of Students who attended the Divinity Hall in the University of Edinburgh, from 1709 to 1727. Copied from the MSS of the Revd. Mr. Hamilton, then Professor of Divinity, etc.* Bayes's name appears in the list for 1720, followed by the letter "*l*," indicating that he was licensed (though not ordained).

6. *List of Theologues in the College of Edin[burgh] since Oct:1711. the 1st. columne contains their names, the 2d the year of their quūmvention, the 3d their entry to the profession, the 4th the names of those who recommend them to the professor, the 5th the bursaries any of them obtain, the 6th their countrey and the 7th the exegeses they had in the Hall.* Here we have

Tho.Bayes|1720|1720|Mr Bayes|—|London|E. Feb. 1721. E. Mar. 1722.

In a further entry in the same volume, in a list headed "Societies," we find Bayes's name in group 5 in both 1720 and 1721. (These were perhaps classes or tutorial groups.) In the list of "Prescribed Exegeses to be delivered" we have

1721. Jan. 14. Mr. Tho: Bayes. the Homily. Matth. 7.24, 25, 26, 27.

and

1722. Ja. 20. Mr. Tho: Bayes. a homily. Matth. 11. 29, 30.

The final entry in this volume occurs in a list entitled "*The names of such as were students of Theology in the university of Edinburgh and have been licensed and ordained since Nov. 1709. Those with the letter .o. after their names are ordained, others licensed only.* Here we find Bayes's name, but without an "o" after it.

There is thus no doubt now that Bayes was educated at Edinburgh University. There is unfortunately no record, at least in those records currently accessible, of any mathematical studies, though he does appear to have pursued logic (under Colin Drummond) and theology.

That Bayes did not take a degree at Edinburgh is in fact not surprising. Grant notes that "after 1708 it was not the interest or concern of any Professor in the Arts Faculty · · · to promote graduation · · · the degree [of Master of Arts] rapidly fell into disregard" [Grant 1884 I, 265]. Bayes was, however, licensed as a preacher, though not ordained.

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NOTES

1. One must not, however, discount the possible effects on the electors of Bayes's *Divine Benevolence* of 1730, a contribution to the Berkleian controversy.

2. The first volume of the *Philosophical Transactions* was for the year 1665.

3. I have omitted here papers on the evaluation of logarithms by series, unless they include *specific* mention of infinite series.

4. This notebook is preserved in the muniment room of the Equitable Life Assurance Society: I am indebted to Mr. H.W. Johnson for providing me with a copy.

5. Some slight attention has been paid to this Notebook by Dale [1986], Holland [1962, 1968], and Home [1974–1975]: a more detailed study is at the moment being undertaken by the present author.

6. The relevant passage in the original runs as follows: Haec autem constans ponendo $x = 1$, quia fit $s = l1 = 0$, ita definietur, ut sit

$$C[\text{onst.}] = 1 - \frac{A}{1.2} + \frac{B}{3.4} - \frac{C}{5.6} + \frac{D}{7.8} - \text{etc.},$$

quae series ob nimiam divergentiam est inepta ad valorem ipsius C[onst.] saltem proxime eruendum. §158. Non solum proximum, sed etiam ipsum verum valorem ipsius C[onst.] inveniemus, si consideremus expressionem Wallisianam pro valore ipsius π inventam atque in *Introductione* demonstratam, quae erat

$$\frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8.10.10.12.\text{etc}}{1.3.3.5.5.7.7.9.9.11.11.\text{etc}}$$

The reference is to Vol. I, Chap. XI, of Euler's *Introductio in Analysin infinitorum*, Lausanne 1748; *opera omnia*, Series I, Vols. 8 and 9.

7. Following Hardy [1949, Sect. 13.2] we defined the Bernoulli numbers B_r by

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \sum_{n=1}^{\infty} (-1)^{n-1} B_n t^{2n}/(2n)!.$$

The first few B_r are

$$B_1 = 1/6; \quad B_2 = 1/30; \quad B_3 = 1/42; \quad B_4 = 1/30; \quad B_5 = 5/66.$$

Various formulae for the constant C in the Euler–MacLaurin sum formula are given in Hardy, *op. cit.* Section 13.13.

The Bernoulli functions $B_n(x)$ are defined by

$$\frac{t\, e^{xt}}{e^t - 1} = 1 + \sum_{n=1}^{\infty} B_n(x)t^n/n!$$

On putting $x = 0$ here and comparing the resulting series with that given above for $t/(e^t - 1)$ one finds that

$$B_{2r}(0) = (-1)^{r-1}B_r; \; B_{2r+1}(0) = 0 \qquad (r > 0).$$

8. To show the equivalence of the series for $\log((z + 1)/z)^{z+1/2}$ given in (3), note first that

$$\Delta^n(1/z) = (-1)^n n!/z(z + 1)\ldots(z + n)$$

and

$$(1/z)^{(n)} = (-1)^n n!/z^{n+1}.$$

Thus

$$\frac{1}{(2r)!}\Delta(1/z)^{(2r-2)} = \frac{1}{(2r)(2r-1)}\Delta(1/z^{2r-1}).$$

This establishes the equivalence of the first two series in (4): that of the second and third series follows from the fact that, for any appropriately differentiable function $f(\cdot)$ of z, $(\Delta f(z))^{(n)} = \Delta f^{(n)}(z)$.

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